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A new estimator method for GARCH models

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Abstract. The GARCH (p, q) model is a very interesting stochastic process with widespread applications and a central role in empirical finance. The Markovian GARCH (1, 1) model has only 3 control parameters and a much discussed question is how to estimate them when a series of some financial asset is given. Besides the maximum likelihood estimator technique, there is another method which uses the variance, the kurtosis and the autocorrelation time to determine them. We propose here to use the standardized 6th moment. The set of parameters obtained in this way produces a very good probability density function and a much better time autocorrelation function. This is true for both studied indexes: NYSE Composite and FTSE 100. The probability of return to the origin is investigated at different time horizons for both Gaussian and Laplacian GARCH models. In spite of the fact that these models show almost identical performances with respect to the final probability density function and to the time autocorrelation function, their scaling properties are, however, very different. The Laplacian GARCH model gives a better scaling exponent for the NYSE time series, whereas the Gaussian dynamics fits better the FTSE scaling exponent.

PACS. 89.65.Gh Economics; econophysics, financial markets, business and management – 05.45.Tp Time series analysis – 89.75.-k Complex systems

1 Introduction

In recent years, physicists have shown an increasing interest in Economics problems [1]. The reason seems to be in the fact that many of these problems may be scrutinized by using standard tools of Statistical Physics [2].

In this paper, we investigate the generalized autoregressive conditional heteroscedasticity (GARCH) model [3–6]. It was designed to describe a central question in theoretical and empirical finance — the volatility. Nowadays, the GARCH model plays an important role in the literature and it may be useful even in the complicated field of accurate forecasts [7–10]. For some model modifications or extensions see [11–17], and for reviews on the subject, see [18–21]

The GARCH (p, q) model has $(p+q+1)$ parameters. Here, we consider only the Markovian process GARCH (1, 1). So, there are three parameters and they can be estimated by evaluating certain quantities of a financial asset. Usually, a Gaussian conditional probability density function is chosen for the GARCH process but many other distributions are possible. A very common method to estimate the GARCH's parameters is the Maximum Likelihood Estimator (mle) [22–24]. An alternative method uses the fact that the variance, the kurtosis and the autocorrelation time are exactly known functions of the parameters. However, in the real world, to get a confident value for the autocorrelation time is very unlikely. Thus, one of the parameters is arbitrarily chosen in order to give a very large autocorrelation time. Although both procedures give reasonable results for the final probability density function (PDF), they fail to reproduce correctly the time dependence of the autocorrelation function. The reason is quite simple: financial assets have autocorrelation function decaying as power law while the Gaussian GARCH decays exponentially [1]. It is worth noticing that we are referring to the autocorrelation time obtained from the autocorrelation function of the square of the return.

Instead of the autocorrelation time, we propose here the use of the standardized 6th moment (i.e., the 6th moment divided by the cubic of the variance). There is an exact formula for this moment in the Gaussian dynamics, but it is only valid within a region which is not relevant for real financial data. For this reason, we propose a perturbation expansion for the 6th moment from which a set of extrapolated parameters of the GARCH model can be calculated. Simulations of the GARCH model, with this set of parameters, reveal a much better agreement with the autocorrelation function of a real asset. Moreover, this characteristic is robust, i.e., it is preserved for both time series studied: NYSE Composite (New York Stock Exchange) and FTSE 100 (Financial Times Stock Exchange).

We also study the Laplacian GARCH model, that is, the GARCH model with the conditional PDF decaying exponentially with the return. We derive an exact expression for the kurtosis, which is written in terms of the GARCH parameters. But, for the sixth moment, an exact formula is unknown. So, we calculate an asymptotic series expansion and then apply our extrapolation method.

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The performances of the Gaussian and the Laplacian GARCH models are compared when applied to the NYSE and FTSE indexes. With respect to the PDF and to the time autocorrelation function, both models are practically equivalent, giving fairly good results. Their differences only appear when the scaling properties are studied. To test the effectiveness of the models, we need to investigate how the probability of return to the origin scales for any time horizon. For the NYSE index, this is the Achilles heel of the Gaussian GARCH model. However, the Laplacian GARCH model works fine. The NYSE index scaling exponent is 0.691 ± 0.062. The Gaussian GARCH exponent is, of course, 0.5, whereas for the Laplacian GARCH, we found 0.635 ± 0.022. So, the Laplacian GARCH agrees, within the error bars, with the real data. On the other hand, the same calculations applied to the FTSE index favors the Gaussian model because the FTSE scaling exponent is 0.49 ± 0.10. In the next section, we present a brief review of the ARCH and GARCH models.

2 ARCH and GARCH models

An ARCH model is a stochastic process with autoregressive conditional heteroscedasticity. It was first introduced by Engle in 1982 [25]. ARCH models are simple models capable to describe a stochastic process which is locally non-stationary but asymptotically stationary. If the stochastic process exhibits a time dependent variance, i.e., volatility, then the ARCH models are particularly useful and therefore have been applied to many different areas of economics: interest rates, stock returns, foreign exchange rates, etc. In an ARCH process, the variance at a time t depends on some past values and it is characterized by a certain number of parameters. An ARCH(p) process is defined by the equation

$$\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + ... + \alpha_p x_{t-p}^2, \tag{1}$$

where the parameters $\alpha_0, \alpha_1, ..., \alpha_p$ are positive constants and x_t is a random variable, with zero mean and variance σ_t^2, coming from some conditional probability density function $P_t(x_t)$. Once the parameters $\alpha' s$ and the form of $P_t(x_t)$ of an ARCH(p) model are chosen, equation (1) is iterated and the asymptotic distribution of x_t is determined and compared with the probability density function of some financial asset. Unfortunately, in order to get good results, ARCH(p) models need very long memories (large p). For this reason, Bollerslev [3] proposed in 1986 a generalization of the ARCH's models, the GARCH (p, q) processes. A GARCH (p, q) model is defined by

$$\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + ... + \alpha_p x_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + ... + \beta_q \sigma_{t-q}^2 \tag{2}$$

here, the $\alpha' s$ and $\beta' s$ are control parameters (all real positive constants) and x_t are random variables with zero mean and variance σ_t^2 obtained from a conditional probability distribution $P_t(x_t)$, usually taken to be Gaussian. In this paper, we restrict our analysis to the Markovian process GARCH (1, 1)

$$\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{3}$$

which has only 3 parameters α_0, α_1 and β_1. The initial condition is assumed to be $\sigma_0^2 = 0$.

There are many methods to estimate the 3 parameters. Perhaps the most common method to fit a GARCH model is based on the maximum likelihood estimator (mle), introduced by Fisher in 1912 [26]. When applied to a financial time series, the mle procedure is as follows [24]. Let $Z(t)$ be the return of some financial asset (see Eq. (13) below). By assuming that the probability distribution of $Z(t)$ is normal, the analysis consists in a numerical run through the whole space of the parameters in order to find the parameters values which maximize the expression

$$\prod_{t=1}^{T} \left[\frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(\frac{-Z^2(t)}{2\sigma_t^2} \right) \right] \tag{4}$$

where σ_t^2 is given by equation (3). There are commercial packages which rapidly determine the GARCH parameters via the mle technique. We used the GARCH toolbox of MATLAB [27].

Another method to estimate the GARCH's parameters involves the calculation of the moments and the time autocorrelation [1,28].

The analytical expression of the n-th moment after T iterations is given by

$$\langle x_T^n \rangle = \int ... \int \prod_{t=1}^{T-1} P_t(x_t) dx_t \int P_T(x_T) x_T^n dx_T. \tag{5}$$

Because the distribution $P_t(x_t)$ has zero mean the variance is equal to the second moment.

The autocorrelation function $\langle x_t x_{t+\tau} \rangle$ of the random variable x_t is proportional to a delta function $\delta(\tau)$. Consequently, only higher-order correlations are interesting or useful. In particular, the autocorrelation for the x_t^2 variable,

$$\langle x_t^2 x_{t+\tau}^2 \rangle = \int ... \int \prod_{t'=1}^{t-1} P_{t'}(x_{t'}) dx_{t'} \int P_t(x_t) x_t^2 dx_t$$
$$\times \int ... \int \prod_{t''=t+1}^{t+\tau-1} P_{t''}(x_{t''}) dx_{t''} \int P_{t+\tau}(x_{t+\tau}) x_{t+\tau}^2 dx_{t+\tau}. \tag{6}$$

From now on, irrespective if we are dealing with real data or simulating GARCH's dynamics, we define the normalized time autocorrelation function $F(\tau)$ of x_t^2

$$F(\tau) = \frac{\langle x_t^2 x_{t+\tau}^2 \rangle - \langle x_t^2 \rangle \langle x_{t+\tau}^2 \rangle}{\langle x_t^4 \rangle - (\langle x_t^2 \rangle)^2}. \tag{7}$$

According to Bollerslev [3], x_t^2 is a Markovian random variable such that

$$\langle x_t^2 x_{t+\tau}^2 \rangle \sim \exp(-\frac{\tau}{\tau_c}), \tag{8}$$

with time autocorrelation

$$\tau_c = |\ln(\alpha_1 + \beta_1)|^{-1}. \tag{9}$$

$$\Theta(\alpha_1,\beta_1)=\frac{15(1+2\alpha_1+2\beta_1+(2+\alpha_1+\beta_1)(\beta_1^2+2\alpha_1\beta_1+3\alpha_1^2))((1-\alpha_1-\beta_1)^2)}{((1-\beta_1^3-3\alpha_1\beta_1^2-9\alpha_1^2\beta_1-15\alpha_1^3)(1-\beta_1^2-2\alpha_1\beta_1-3\alpha_1^2))} \tag{19}$$

The equation above, together with the 2th and 4th moments, compound a second method that we shall refer as τ_c.

Here we propose a new method to estimate the GARCH's parameters. The parameters α_1 and β_1 are evaluated using, simultaneously, the exact kurtosis expression and an asymptotic series expansion of the sixth moment. The parameter α_0 is then obtained from the variance formula. Considering the NYSE and the FTSE indexes, we shall show that our prescription gives better results than the mle and the τ_c methods.

3 Gaussian GARCH models

A model is said a Gaussian GARCH model if the conditional probability function $P_t(x_t)$ is Gaussian with fluctuating variance. More explicitly

$$P_t(x_t)=\frac{\exp(-\frac{x_t^2}{2\sigma_t^2})}{\sigma_t(2\pi)^{\frac{1}{2}}}, \tag{10}$$

where the variance σ_t^2 change with time according equation (3).

Using equations (3, 5) and (10), it can be proved [29] that the variance σ_t^2 converges to

$$\sigma^2(\alpha_0,\alpha_1,\beta_1)=\frac{\alpha_0}{1-\alpha_1-\beta_1}, \tag{11}$$

inasmuch as $(\alpha_1+\beta_1)<1$. Indeed, with this restriction, equation (11) holds for any probability density function.

The kurtosis κ is defined as the 4th moment divided by the square of the variance. For a Gaussian distribution, its exact expression is given by [29]

$$\kappa_G(\alpha_1,\beta_1)=\frac{\langle x^4\rangle}{\sigma^4}=3+\frac{6\alpha_1^2}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}. \tag{12}$$

In the τ_c method, the analytic equations (9, 11) and (12) are made equal to the autocorrelation time, the variance and the kurtosis calculated from the time series of some financial asset. As we shall see, the great obstacle is the autocorrelation time τ_c. Its determination is very difficult and imprecise. The point is that, in the real world, ensemble averages are not possible so a good estimation of τ_c is out of question.

We analyze the daily values of the NYSE composite index, recorded from December 31, 1965 to January 31, 2006 (10 088 points) and the FTSE 100 index, recorded from April 2, 1984 to January 30, 2007 (5 768 points). Let $Y(t)$ be the index value at time t. As our frequency data is very low, the recommended random variable to be used here is the return $Z_{\Delta t}(t)$ defined by

$$Z_{\Delta t}(t)=\ln(Y(t+\Delta t))-\ln(Y(t)) \tag{13}$$

hereafter, for the sake of simplicity, we will denote $Z(t)\equiv Z_{\Delta t=1}(t)$ when $\Delta t=1$.

In the case of the maximum likelihood estimator, we used the GARCH toolbox of MATLAB to determine the control parameters for the NYSE and FTSE time series

$$s_{mle}^{nyse}=(\alpha_0,\alpha_1,\beta_1)=(1.23\times10^{-6},0.080,0.906) \tag{14}$$
$$s_{mle}^{ftse}=(\alpha_0,\alpha_1,\beta_1)=(1.69\times10^{-6},0.089,0.894). \tag{15}$$

For the NYSE (FTSE) index, the variance and kurtosis can be immediately determined: $\sigma_{nyse}^2=8.084\times10^{-5}$ (1.035×10^{-4}) and $\kappa_{nyse}=38.507$ (11.061). On the other hand, the time autocorrelation function $F(\tau)$ has the very wild behavior shown in Figure 3. An estimative of τ_c can be obtained by finding that time in which $F(\tau)$ turns negative for the first time. We get $\tau_c^{nyse}=143$ (224).

Substituting the values above into equations (9, 11) and (12), we calculate the GARCH set of parameters s_{τ_c}

$$s_{\tau_c}^{nyse}=(\alpha_0,\alpha_1,\beta_1)=(5.63\times10^{-7},0.080,0.913) \tag{16}$$
$$s_{\tau_c}^{ftse}=(\alpha_0,\alpha_1,\beta_1)=(4.56\times10^{-7},0.057,0.939). \tag{17}$$

Values of β_1, bigger than 0.9, are often used in the literature, e. g., $\beta_1=0.90000$ for the S&P 500 [1] and $\beta_1=0.90501$ for the stock prices of the Center for Research in Security Prices (CRSP) [28].

Instead of using the maximum likelihood estimator or the time autocorrelation τ_c, we propose the series expansion of the *standardized* 6th moment Θ as a new way to estimate the GARCH control parameters. It is defined as the 6th moment divided by the cube of the variance

$$\Theta(\alpha_1,\beta_1)=\frac{\langle x^6\rangle}{\sigma^6}. \tag{18}$$

This is a natural choice since it is the next even moment after the kurtosis.

In his paper, Bollerslev [3] derived an analytical expression for all even moments of a GARCH process. The existence of these moments holds in a very limited region of the parameters space (α_1,β_1). The higher the moment, smaller the region. For the standardized sixth moment, we have the exact formula

see equation (19) above.

Financial indexes have time increasing moments. The higher the moment, the faster is its divergence. In the Bollerslev's paper [3], the GARCH process is assumed to be stationary in the wide-sense, that is, its moments are time independent. However, financial data cannot be described by a wide-sense stationary stochastic process, they are at best asymptotically stationary! This means that, for real financial markets, the GARCH parameters are beyond the radius of convergence of the exact formula given in (19). Indeed, if we substitute the α_1,β_1 values estimated by the mle and τ_c techniques into the equation (19), we get *negative* moments: $(-2,131.3;-178.5)$ for the NYSE and $(-644.3;-108.0)$ for the FTSE index, respectively.

We can derive a series expansion for the sixth moment by substituting equations (3) and (10) into equation (5). This expression can be integrated for a given number of iterations T. The result is α_0^3 times a two-variable polynomial in α_1 and β_1. This means that $\Theta(\alpha_1,\beta_1,T)$ is a T-sequence of polynomials functions of α_1 and β_1. The powers of these polynomials grow up with T. For example, for $T=3$ and $T=4$ we have

$$\begin{aligned}\Theta(\alpha_1,\beta_1,3) &= 15+90\alpha_1^2+120\alpha_1^3+270\alpha_1^4+180\beta_1\alpha_1^3\\ &+90\beta_1^2\alpha_1^2+1080\alpha_1^5+360\alpha_1^4\beta_1+1800\alpha_1^6+360\alpha_1^4\beta_1^2\\ &+1080\alpha_1^5\beta_1+120\alpha_1^3\beta_1^3 \end{aligned}\quad (20)$$

and

$$\begin{aligned}\Theta(\alpha_1,\beta_1,4) &= 15+90\alpha_1^2+120\alpha_1^3+270\alpha_1^4+180\beta_1\alpha_1^3\\ &+90\beta_1^2\alpha_1^2+1080\alpha_1^5+360\alpha_1^4\beta_1+2610\alpha_1^6+1260\alpha_1^4\beta_1^2\\ &+2160\alpha_1^5\beta_1+480\alpha_1^3\beta_1^3+...+27000\alpha_1^9. \end{aligned}\quad (21)$$

One can verify that, up to order $(m+n)=5$, the coefficients of $\alpha_1^m\beta_1^n$ in $\Theta(\alpha_1,\beta_1,3)$ remain unaltered for all further iterations ($T\geq 4$), so one can write down an asymptotic series expansion

$$\begin{aligned}\bar{\Theta}(\alpha_1,\beta_1,3) = &\ 15+90\alpha_1^2+120\alpha_1^3+270\alpha_1^4+180\beta_1\alpha_1^3\\ &+90\beta_1^2\alpha_1^2+1080\alpha_1^5+360\alpha_1^4\beta_1. \end{aligned}\quad (22)$$

In general, the $\bar{\Theta}(\alpha_1,\beta_1,T)$ series expansion in $\alpha_1^m\beta_1^n$ is *exact* up to order $(m+n)=2T-1$.

In fact, this expansion is *exactly the same* as the one arising from a Taylor expansion of formula (19).

Naturally, the later procedure is easier than ours, which starts from the definition and requires several integrations. However, it is the only possible way when no exact sixth moment formulas are available (see, for example, the next section).

Using Maple [30], we were able to go up to $T=30$. Thus there is a sequence of $\bar{\Theta}(\alpha_1,\beta_1,T)$, $T=1,\ldots,30$, which can be explored by extrapolation techniques. For the NYSE (FTSE) series, we have $\kappa_{nyse}=38.507$ (11.061) and $\Theta_{nyse}=17717.37$ (888.35). Solving simultaneously the kurtosis equation $\kappa_G(\alpha_1,\beta_1)$ and $\bar{\Theta}_G(\alpha_1,\beta_1,T)$ for diverse T, we obtain a sequence of curves $\beta_1(\alpha_1,T)$ plotted in Figure 1. Where these curves cross the kurtosis curve, gives a sequence of solutions $\alpha_1(T)$ which are shown in the insets. The data are very well fitted by the exponential

$$\alpha_1(T)=B\exp(-\frac{T}{\Gamma})+C, \quad (23)$$

where, for the NYSE (FTSE) $B=0.82\pm0.04$, (0.48 ± 0.02) $\Gamma=9.69\pm0.52$ (11.13 ± 0.64) and $C=0.246\pm0.006$ (0.184 ± 0.004). So, in the limit $T\to\infty$, the extrapolated value of α_1 are 0.246 (0.184). Using these values in the equations (11) and (12), we obtain α_0 and $\beta 1$. The new sets s_{6th} for the control parameters are

$$s_{6th}^{nyse}=(\alpha_0,\alpha_1,\beta_1)=(5.49\times10^{-6},0.246,0.686) \quad (24)$$
$$s_{6th}^{ftse}=(\alpha_0,\alpha_1,\beta_1)=(0.49\times10^{-5},0.184,0.768) \quad (25)$$

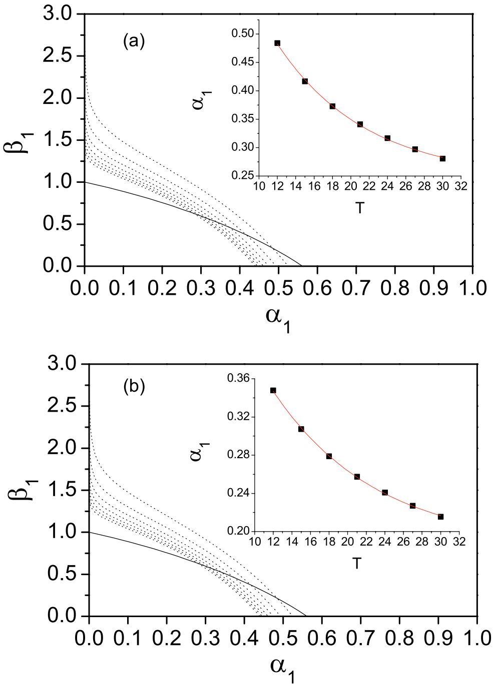

Fig. 1. (a) NYSE — Full line is the solution of the kurtosis equation $\kappa_G(\alpha_1,\beta_1)=38.507$. Dotted lines are the solutions of the standardized 6th moment sequence $\bar{\Theta}_G(\alpha_1,\beta_1,T)=17,717.37$ for $T=12,15,18,21,24,27,30$. The crossing points generate a sequence $\alpha_1(T)$ plotted in the inset. (b) FTSE — the same as (a) but with $\kappa_G(\alpha_1,\beta_1)=11.061$ and $\bar{\Theta}_G(\alpha_1,\beta_1,T)=888.35$

We can now simulate Gaussian GARCH processes using the three sets of parameters. In Figure 2, we plot the PDF of the NYSE (FTSE) index together with the 10 088 (5 768) points generated by GARCH's dynamics, averaged in an ensemble of 10 000 runs, for the sets s_{mle}, s_{τ_c} and s_{6th}.

In order to check out the adjusting quality of the simulated PDF, we evaluate the Pearson's chi-square test [31]. Let us denote by $H(i)$ the *histogram* value at point i, then

$$\chi_s^2=\sum_{i=1}^{M}\frac{(H_{index}(i)-H_s(i))^2}{(H_{index}(i)+H_s(i))}, \quad (26)$$

where M is the total number of graph points and s is one of the parameter sets. We found for the NYSE (FTSE) the following values: $\chi_{s_{mle}}=206.4$ (70.4), $\chi_{s_{\tau_c}}=71.7$ (67.9) and $\chi_{s_{6th}}=93.3$ (65.6). As we can see, the mle method gives the worst results for both indexes. On the other hand, our technique works slightly better for the FTSE index, whereas the τ_c procedure do the same for the NYSE index.

What can we say about the autocorrelation function $F(\tau)$? Which set of parameters gives the best results? It is clear from the Figure 3a that the set s_{6th} is much superior than the others. The results for the sets τ_c are the worst for both indexes. In the case of the FTSE index, they are almost equivalent.

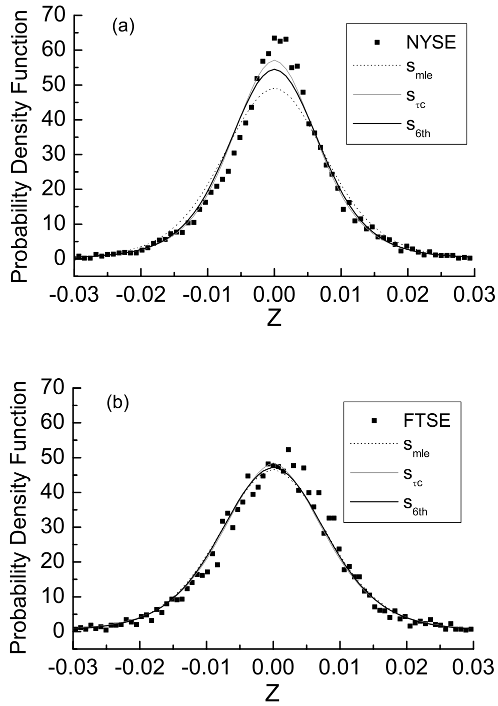

Fig. 2. (a) The probability density function for 10,088 points of the NYSE composite index. (b) The probability density function for 5 768 points of the FTSE 100 index.

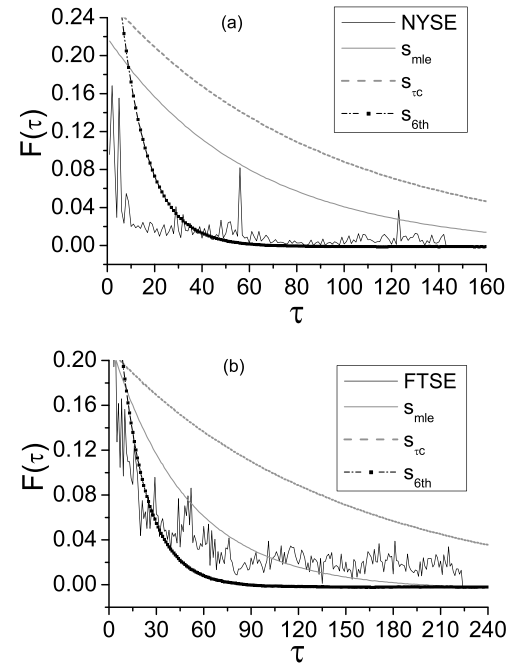

Fig. 3. The autocorrelation function $F(\tau)$ versus time τ. (a) NYSE index, (b) FTSE index. Simulations were carried out over an ensemble of 10 000 runs.

Taking into account the results for the histograms and the autocorrelation functions, we conclude that our proposal gives the best fitting.

As it is well-known, the Gaussian GARCH dynamics is unable to describe the time scaling properties of a financial index. For this reason, in the next section, we apply our 6th-moment technique to the GARCH (1, 1) model with Laplacian conditional probability [16].

4 Laplacian GARCH models

An alternative to the Gaussian conditional probability is the Laplacian, defined as

$$P_t(x_t) = \frac{\exp(-\frac{\sqrt{2}\,|x_t|}{\sigma_t})}{\sigma_t\sqrt{2}}, \tag{27}$$

where the variance σ_t^2 changes with time according to equation (3). As we shall see, the Laplacian gives a more leptokurtic PDF.

Using the Laplacian distribution, we investigate the moments which can be derived from equation (5). The second moment is given once more by the equation (11), but the kurtosis relation is quite different. We deduced the following analytic expression for the kurtosis of a Laplacian GARCH process

$$\kappa_L(\alpha_1, \beta_1) = \frac{\langle x^4 \rangle}{\sigma^4} = 6 + \frac{30\alpha_1^2}{1 - 6\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2}. \tag{28}$$

Due to the fact that an exact formula for the standardized 6th moment $\Theta_L(\alpha_1, \beta_1)$ is not known, we follow our procedure described in the previous section. We substitute equations (3) and (27) into (5), integrating the result up to some given order T. Once again, this series turns to be *exact* up to order $(2T - 1)$.

Utilizing the software Maple, we were able to go up to $T = 18$. The sequence of solutions $\alpha_1(T)$ are very precisely fitted by the exponential of the equation (23). For the NYSE (FTSE) we got $B = 0.40 \pm 0.01$, (0.20 ± 0.01) $\Gamma = 6.77 \pm 0.26$ (8.02 ± 0.35) and $C = 0.138 \pm 0.003$ (0.097 ± 0.002). The extrapolated values of α_1 are thus 0.138 and 0.097 (see Fig. 4). Using equations (11) and (28), we arrive to the new sets ss_{6th} for the control parameters

$$ss_{6th}^{nyse} = (\alpha_0, \alpha_1, \beta_1) = (4.73 \times 10^{-6}, 0.138, 0.803) \tag{29}$$

$$ss_{6th}^{ftse} = (\alpha_0, \alpha_1, \beta_1) = (5.49 \times 10^{-6}, 0.097, 0.850). \tag{30}$$

In the next section, based on our 6th moment series expansion method, we will try to find out which dynamics, Gaussian or Laplacian, gives better results for the time scaling properties of both indexes.

5 Time scaling properties

First, let us compare the capabilities of a Gaussian and an Laplacian GARCH process to describe the PDF's and the time autocorrelation function using the set of parameters coming from the standardized 6th moment series technique.

In Figure 5, we present the NYSE and FTSE PDF's for both GARCH dynamics. The calculated χ^2 deviations for the *histograms* of the NYSE (FTSE) index are $\chi^2 = 93.3$

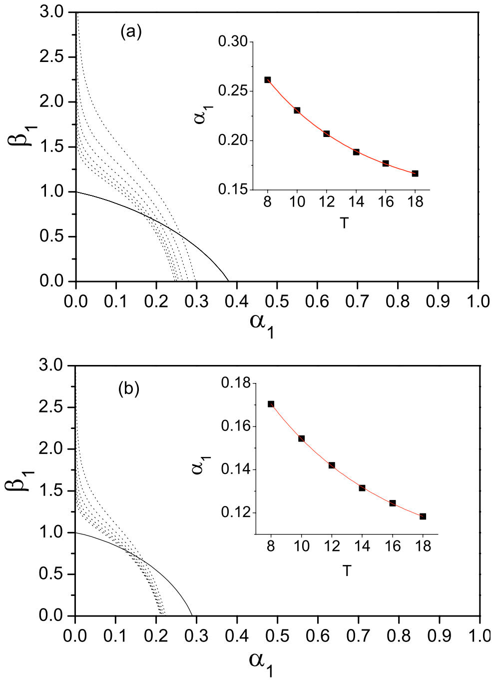

Fig. 4. (a) NYSE — Full line is the solution of the kurtosis equation $\kappa_L(\alpha_1, \beta_1) = 38.507$. Dotted lines are the solutions of the standardized 6th moment sequence $\bar{\Theta}_L(\alpha_1, \beta_1, T) = 17,717.37$ for $T = 8, 10, 12, 14, 16, 18$. The crossing points generate a sequence $\alpha_1(T)$ plotted in the inset. (b) FTSE — the same as (a) but with $\kappa_L(\alpha_1, \beta_1) = 11.061$ and $\bar{\Theta}_L(\alpha_1, \beta_1, T) = 888.35$.

(65.6) and $\chi^2 = 189.2$ (205.9) for the Gaussian and Laplacian dynamics, respectively. So, the Gaussian GARCH is closer to the NYSE and FTSE PDF's.

Figure 6 plots the time autocorrelation function $F(\tau)$. In the NYSE case (Fig. 6a), a mere visual inspection is sufficient to conclude that the Laplacian formalism is superior for all times. However, for the FTSE index (Fig. 6b),the Gaussian GARCH is slightly better than the Laplacian (mainly in the interval $30 < \tau < 70$). Below, we investigate the time scaling properties of the two dynamics.

By using equation (13), we can determine, for each one of them, the probability of return to the origin $P(0)$. This quantity scales as a power law

$$P(0) \sim (\Delta t)^{-\alpha}. \tag{31}$$

By changing the time horizon Δt, one can answer whether the overall dynamics is well described by a GARCH (1, 1) process.

In Figure 7a we plot $P(0)$ versus Δt for the NYSE series. We find the following exponents: $\alpha = 0.69 \pm 0.07$ (NYSE), $\alpha = 0.499 \pm 0.001$ (Gaussian) and $\alpha = 0.63 \pm 0.03$ (Laplacian). This latter value is very near to the NYSE exponent and within the error bars.

In Figure 7b, we show the results for the FTSE index. We get the exponents: $\alpha = 0.49 \pm 0.10$ (FTSE), $\alpha = 0.499 \pm 0.001$ (Gaussian) and $\alpha = 0.65 \pm 0.03$ (Laplacian). In contrast to the NYSE case, the FTSE index is better described by the Gaussian dynamics.

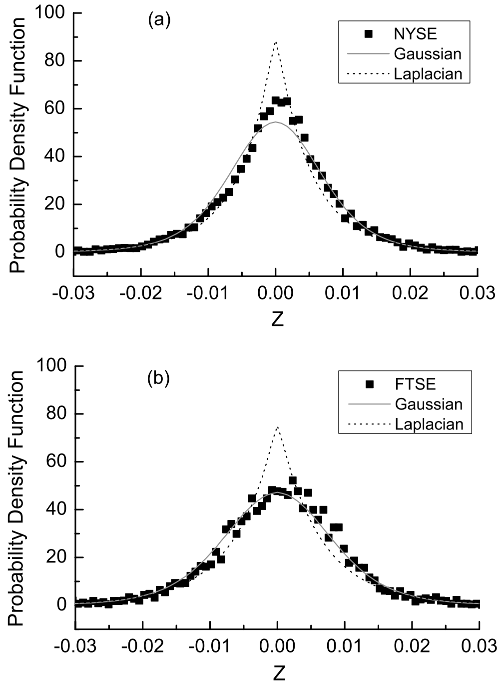

Fig. 5. The PDF's versus the returns Z. The Gaussian as well as the Laplacian GARCH PDF's were obtained by using the set of parameters s_{6th} and ss_{6th}.

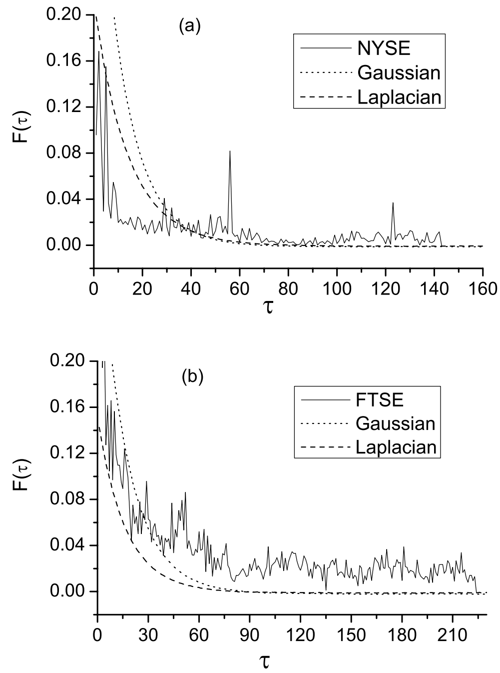

Fig. 6. The time autocorrelation function $F(\tau)$ versus time τ for the NYSE and FTSE indexes.

6 Conclusions

We conclude that, in spite of their wide use in the literature, the maximum likelihood estimator and the time autocorrelation methods do not generate the best set of GARCH parameters. In our paper, we propose to estimate the GARCH parameters through a perturbative expansion

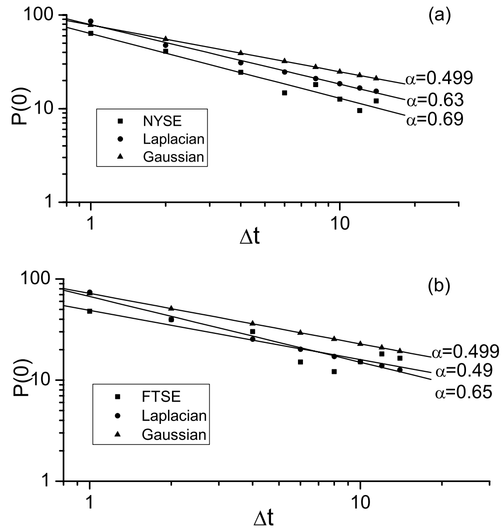

Fig. 7. The probability of return to the origin versus the time horizon Δt for the NYSE and the FTSE indexes.

of the sixth moment. We extrapolate the sequence of crossing points of the kurtosis curve with the sixth moment asymptotic expansion up to some given and small order T. We think this a good way to treat the non-stationarity problem. As the extrapolated value of the α_1 parameter has an error smaller than 1% (for all cases described in our paper), we believe that our perturbation method is well justified.

Our proposal of using the standardized 6th moment asymptotic series expansion produces a set of parameters s_{6th} which gives a nice fit for the probability density function and a much better adjust for the autocorrelation function. Moreover, this result is robust since it is valid for both indexes studied: the NYSE composite and the FTSE 100. Last but not least, we observe that our method is applicable to any conditional probability function.

Despite all these accomplishments, our method is more involved than the others. There is a laborious series expansion and extrapolation which certainly demand more CPU time. In order to set error bars to our 6th moment estimator, we tested our method on a simulated Gaussian GARCH time series. Averaging over 10^6 realizations, we got for the NYSE index, $s_{6th} = (\alpha_0, \alpha_1, \beta_1) = (5.49 \times 10^{-6} \pm 4.61 \times 10^{-6}, 0.24 \pm 0.02, 0.68 \pm 0.03)$.

Based on our 6th moment series expansion method, we studied the performance of the GARCH (1, 1) model under two different conditional probabilities: Gaussian and Laplacian distributions. For the Laplacian case, we derived an exact expression of the kurtosis as a function of the control parameters.

A comparison between the Gaussian and Laplacian forms shows that the Gaussian GARCH gives a better PDF for both indexes. However, with respect to the autocorrelation function, the Laplacian formalism fits better the NYSE index while the FTSE is finer adjusted by the Gaussian dynamics.

Finally, when the probability of return to the origin of the NYSE index is analyzed, the Laplacian GARCH performs much better than its Gaussian counterpart. On the other hand, the Gaussian formalism gives the finest time scaling exponent to the FTSE index.

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